CHAPTER 1

Topology

BY

Dr. Dibyendu De

Associate Professor

Department of Mathematics

University Of Kalyani

West Bengal, India

E-mail : dibyendude@gmail.com

Module-2: Base of Topological spaces

Since open sets are defined in terms of open intervals many arguments with open sets in $\mathbb{R}$ reduce to looking at what happens with open intervals. A similar statement holds for $\mathbb{R}^n$ with open balls in place of open intervals. In each case arbitrary open sets are unions of the special open sets given by open intervals, and balls. Generalizing this idea we introduce the following definition.

Definition 1. *Let X be a nonempty set. A collection $\mathcal{B}$ is said to be a basis for some topology on X if $\mathcal{B}$ satisfies the following two properties:*

(1) Every point $x \in X$ lies in some set $B \in \mathcal{B}$.

(2) For each pair of sets B_1, B_2 in $\mathcal{B}$ and each point $x \in B_1 \cap B_2$ there exists a set B_3 in $\mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$.

The topology generated by a basis $\mathcal{B}$, generally denoted by $\tau(\mathcal{B})$ can be defined as follows :

A subset $O \subset X$ is to be declared as open if for any $x \in O$ there exists some $B \in \mathcal{B}$ such that $x \in B \subset O$.

It needs to prove that this $\tau(\mathcal{B})$ is in fact a topology.

Theorem 1. *Let X be a non empty set and $\mathcal{B}$ be a basis. Let $\tau(\mathcal{B})$ be the collection defined as follows: $U \in \tau(\mathcal{B})$ if for each $x \in U$ there exists some $B \in \mathcal{B}$ with the property $x \in B \subset U$. Then $\tau(\mathcal{B})$ is a topology on X.*

Proof. Vacuously $\emptyset \in \tau(\mathcal{B})$ and $X \in \tau(\mathcal{B})$ is obvious.
That $\tau(\mathcal{B})$ is closed under arbitrary union is also clear.
Finite intersection property follows from point (2) of definition of basis.

□

Most known example is that $\mathbb{R}$ with usual topology has the following two bases:

1. $\{(a, b) : a, b \in \mathbb{R}\}$
2. $\{(a, b) : a, b \in \mathbb{Q}\}$.

Another way of describing the topology generated by a basis is given in the following lemma:

Lemma 1. *Let X be a set; let $\mathcal{B}$ be a basis on X. Then $\tau(\mathcal{B})$, the topology generated by $\mathcal{B}$ equals the collection of all unions of elements of $\mathcal{B}$.*

Proof. Since $\tau(\mathcal{B})$ is a topology generated by $\mathcal{B}$, all possible union of members of $\mathcal{B}$ is a subset of $\tau(\mathcal{B})$. Conversely, given $U \in \tau(\mathcal{B})$ choose for each $x \in U$ an element B_x of $\mathcal{B}$ such that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$, so U equals a union of elements of $\mathcal{B}$. □

Observation 1. *The above theorem shows that a topology on a set is all possible unions of members of base.*

In the above Theorem we mentioned how to from topology from a basis. The following is one way of obtaining a basis for a given topology. We shall use it frequently.

Proposition 1. *Let $\mathcal{B}$ is a collection of open sets of a topological space X satisfying that for each open set U of X and each x in U, there is an element in $B \in \mathcal{B}$ such that $x \in B \subset U$. Then $\mathcal{B}$ is base for the topological space X.*

Proof. The first condition, is obvious. For the second condition, suppose B_1 and B_2 are elements of $\mathcal{B}$ and $x \in B_1 \cap B_2$. Since $B_1 \cap B_2$ is an open set there exists some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.
The topology generated by $\mathcal{B}$ equals to the topology of X, is left as an exercise.

□

We have already seen that both the following sets generates the usual topology on $\mathbb{R}$.

1. $\{(a, b) : a, b \in \mathbb{R}\}$
2. $\{(a, b) : a, b \in \mathbb{Q}\}$. But when topologies are given by bases, it is useful to have a criterion in terms of the bases for determining whether one topology is finer than another.

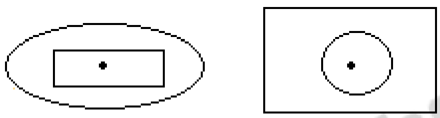
Proposition 2. *Let $\mathcal{B}$ and $\mathcal{B}_1$ for the topologies τ_1 and τ_2 on a set X. Then the following conditions are equivalent.*

(1) τ_2 is finer than τ_1.
(2) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subset B_1$.

Proof. 1) $\Rightarrow$ 2). Given an element U of τ_1, we wish to show that U is in τ_2 also. Let $x \in U$. Since $\mathcal{B}_1$ generates τ_1 there is an element $B \in \mathcal{B}$ such that $x \in B \subset U$. Condition 2) tells us there exists an element $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subset B_1$. Then $x \in B_2 \subset U$,so U is in τ_2 by definition.

2) $\Rightarrow$ 1). We are given $x \in X$ and $B \in \mathcal{B}$, with $x \in B$. Now B belongs to τ_1 by definition and $\tau_1 \subset \tau_2$. by condition 1); therefore, $B \in \tau_2$ Since τ_2 is generated by $\mathcal{B}_2$ there is an element $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subset B_1$.

$\square$



Example 1. *1. One of the most beautiful example of the above Theorem is that the set $\{(a, b) : a, b \in \mathbb{R}\}$ and $\{(a, b) : a, b \in \mathbb{Q}\}$ both generates the same topology $\mathbb{R}$. It happens as $\mathbb{Q}$ is dense in $\mathbb{R}$.*
2. Another beautiful application of the above Theorem is that open rectangles in $\mathbb{R}^2$ and open discs in $\mathbb{R}^2$ generates the same topology on $\mathbb{R}^2$. In fact if we take an open disk D and a point x in D then we can inscribed a rectangle in D containing x. Similarly the other. Therefore the topologies generated by them are equivalent.

Example 2. *Sorjenfrey line is an extremely important topological space. This is alternatively known as lower limit topology.*

Consider the collection $\mathcal{B} = \{[a, b) : a, b \in \mathbb{R}\}$ of subsets of $\mathbb{R}$. Then it is a basis for some topology on $\mathbb{R}$. In fact
a. For any $r \in \mathbb{R}$, $r \in [r, r + 1)$,
b. If $[a, b)$ and $[c, d)$ be two members of $\mathcal{B}$ and $r \in [a, b) \cap [c, d)$ then clearly the intersection is in $\mathcal{B}$.

Therefore the collection $\mathcal{B}$ generates some topology on $\mathbb{R}$, which is known as lower limit topology and denoted by $\mathbb{R}_l$.

An interesting point to observe that, $\mathbb{R}_l$ has a basis each of whose member is closed as well as open. In fact each set $[a, b)$ is also closed in $\mathbb{R}_l$. In fact if $x \notin [a, b)$ then either $x < a$ or $x \geq b$. In the first case we can choose $r \in (x, a)$ sot that $[x, a)$ misses $[a, b)$ and in the second case $[x, x + 1)$ misses $[a, b)$. Therefore x is not a limit point of $[a, b)$ and hence $[a, b)$ is a closed set.

Another important observation about $\mathbb{R}_l$ is that each open interval in $\mathbb{R}_l$ is open, that means that the topology of $\mathbb{R}_l$ is larger than the usual topology.

Example 3. *Show that the collection*

$$\mathcal{B} = \{[x, y) : x < y \text{ and } x, y \in \mathbb{Q}\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

Example 4. *Consider the set $K = \{\frac{1}{n} : n \in \mathbb{N}\}$ and the subsets of the form $(a, b) \setminus K$. The collection*

$$B_1 = \{(a, b) \subset \mathbb{R} : a, b \in \mathbb{R}\} \cup \{(a, b) \setminus K \subset \mathbb{R} : a, b \in \mathbb{R}\}$$

is a basis for a topology on $\mathbb{R}$: The topology it generates is known as the K-topology on $\mathbb{R}$: Clearly, K-topology is finer than the usual topology. Note that there is no neighbourhood of 0 in the usual topology which is contained in $(-1, 1) \setminus K \in \mathcal{B}$. This shows that the usual topology is not finer than K-topology. The same argument shows that the lower limit topology is not finer than K-topology. Consider next the nbd $[2, 3)$ of 2 in $\mathbb{R}_l$. Then there is no nbd of 2 in the K-topology which is contained in $[2, 3)$. Thus we conclude that the K-topology and the lower limit topology are not comparable.

A partially ordered set is a pair $(X, \leq)$, where X is a set and $\leq$ is a relation on X such that: (i) $x \leq x$ for all x; (ii) if $x \leq y$ and $y \leq z$, then $x \leq z$; (iii) if $x \leq y$ and $y \leq x$, then $x = y$. Let us call a partially ordered set linearly ordered if whenever x and y are in X, either $x \leq y$ or $y \leq x$. (For example, $X = \mathbb{R}$ or any of its subsets is linearly ordered.) (a) If X is a partially ordered set and $\mathcal{S}$ is the collection of all sets having the form of either

$\{y : y \leq x \text{ and } y \neq x\}$ or $\{y : x \leq y \text{ and } y \neq x\}$, we can show that $\mathcal{S}$ is a subbase for a topology on X. This is called the order topology on X. (b) Show that if $(X, \leq)$ is linearly ordered, then the order topology satisfies the Hausdorff property. Can you find another condition on the ordering such that the order topology has the Hausdorff property? (c) When a and b are elements of a partially ordered space, let $(a, b) = \{x \in X : a < x < b\}$. If $(X, \leq)$ is linearly ordered, show that $(a, b) = \{x \in X : a < x < b\}$ is a base of the order topology.

Definition 2. *A neighborhood of a point x in a topological space X is any set $A \subset X$ that contains an open set O containing x. Dually x is said to be an interior point of A; that is, A is a neighbourhood of x if and only if $x \in A^\circ$. The collection $\mathcal{A}_x$ of all neighbourhoods of x is the neighbourhood system of x.*

Proposition 3. *The neighbourhood system $\mathcal{A}_x$ at x in a topological space X has the following properties:*

(a) $\mathcal{A}_x \neq \emptyset$, for all $x \in X$

(b) if $A \in \mathcal{A}_x$ then $x \in A$

(c) if $A_1, A_2 \in \mathcal{A}_x$ then $A_1 \cap A_2 \in \mathcal{A}_x$

(d) if $A \in \mathcal{A}_x$ then there is a $B \in \mathcal{A}_x$ such that $A \in \mathcal{A}_y$ for each $y \in B$

(e) if $A \in \mathcal{A}_x$ and $A \subset B$ then $B \in \mathcal{A}_x$.

$O \subset X$ is open if and only if O contains a neighbourhood of each of its points. Show that this conditions generates a unique topology on X.

Proof. Let τ be the collection of all such sets in X. Vacuously $\emptyset \in \tau$ and $X \in \tau$ is obvious. Let $A_1, A_2 \in \tau$ and $x \in A_1 \cap A_2$. Then there exists $O_1, O_2 \in \mathcal{A}_x$ such that $x \in O_i \subset A_i$ for all i. So by condition (c) $A_1, A_2 \in \tau$. Arbitrary union of membered of τ is in τ is obvious. Hence τ is a topology.

□

As an example of this property consider the set of open balls in a metric space.

For $a \in \mathbb{Z}$ and $K \in \mathbb{N}$, we define $a + K\mathbb{Z} = \{a + Kn \colon n \in \mathbb{Z}\}$. We say that $A \subset \mathbb{Z}$ is open if for each point $a \in A$, there is a number $K \in \mathbb{N}$ such that $\{a + Kn \colon n \in \mathbb{Z}\} \subset A$. In other words, a subset of $\mathbb{Z}$ is open if each of its points is contained in an arithmetic progression belonging to the set. Obviously the sets $\{a + Kn \colon n \in \mathbb{Z}\}$ are open. So the arithmetic progressions form a basis for the topology. Surprisingly, this basic sets are closed also because the complement of $a + K\mathbb{Z} = \{a + Kn : n \in \mathbb{Z}\}$ is the union of other arithmetic progressions with the same difference.

There are many proofs that there exist an infinity of primes. Using this topology we can prove that the infiniteness of primes.

Proposition 4. *The number of primes is infinite.*

Proof. Since the sets $\{i, i+d, i+2d, ...\}$, $i = 1, 2, ..., d$ are open, pairwise disjoint and cover the whole $\mathbb{N}$, it follows that each of them is closed. In particular, for each prime number p the set $\{p, 2p, 3p, ...\}$ is closed. All together, the set of the form $\{p, 2p, 3p, ...\}$ cover $\mathbb{N} - \{1\}$. Hence if the set of prime numbers were finite, then the set $\{1\}$ would be open. However, it is not a union of arithmetic progressions. $\square$

Proposition 5. *Let X be a set. Given any family $\mathcal{S} = \{S_\alpha : \alpha \in I\}$ with $X \subset \cup_\alpha S_\alpha$. of subsets of X, there always exist a unique smallest topology generated by $\mathcal{S}$. $\mathcal{S}$ is called a subbasis for $\tau(\mathcal{S})$, the topology generated by $\mathcal{S}$.*

Proof. Let $\tau(\mathcal{S})$ be the intersection of all topologies containing $\mathcal{S}$; such topology exists, since $\mathcal{P}(X)$ is one such. So clearly $\tau(\mathcal{S})$ is a topology. It evidently satisfies the requirements of 'unique' and 'smallest'. To verify the members of $\tau(\mathcal{S})$ are as described, note that since $\mathcal{S} \subset \tau(\mathcal{S})$ so $\tau(\mathcal{S})$ must contain all the sets listed. Conversely, since $\bigcup_\alpha$ distributes over $\bigcap$, the sets listed actually do form a topology containing $\mathcal{S}$, and which therefore contains $\tau(\mathcal{S})$. $\square$

Remark 1. *The construction of a topology from a subbasis loses some control over the open sets; they build up from the finite intersections of the S_α's rather than from the S_α themselves.*

Example 5. *Every basis for a topology is a subbasis.*
(a)$\mathcal{S} = \{(a, \infty), (-\infty, b) : a, b \in \mathbb{R}\}$ is a subbasis for the usual topology on $\mathbb{R}$. We can restrict the numbers a in these intervals to be rational or irrational, and we still have a subbasis that generates the usual topology.

(b) $\mathcal{S} = \{[a, \infty), (-\infty, b) : a, b \in \mathbb{R}\}$ is a subbasis for $\mathbb{R}_l$

(c) $\mathcal{S} = \{\mathbb{R} - \{a\} : a \in \mathbb{R}\}$ is a subbasis for the cofinite topology on $\mathbb{R}$.